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SLIDING OF A CYLINDER ON A VISCOELASTIC FOUNDATION

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A method is proposed for the solution of the contact problem concerning the motion of a stamp on a viscoelastic foundation. Applied to a stamp of cylindrical shape, the method yields an extension of the results obtained for a standard linear body to the case of a discrete relaxation time spectrum.

In attempts to construct a quantitative theory of the friction of solids, the friction force is ordinarily separated into deformation and molecular components [1]. The deformation component is determined by the viscoelastic and plastic properties of the bodies making contact. Elastic processes play the main part in steady motion, while the deformation component of the friction is determined by hysteresis losses during the re-deformation of the material [1, 2]. The hysteresis phenomena are associated with the nonequilibrium of the deformation process for viscoelastic solids, and the degree of nonequilibrium of the process depends on the relationship between the characteristic times the relaxation processes occurring in the medium and the contact time of a given point of the medium surface with unit roughness. Therefore, the solution of the contact problem concerning the sliding of a stamp on a viscoelastic foundation affords the possibility of determining the nature of the velocity dependence on friction. The case of a cylindrical stamp, for which the investigation is facilitated considerably in connection with the two-dimensionality of the problem, is taken below as a specific example.

The question of the sliding of a cylinder on a viscoelastic foundation has been examined earlier in a number of papers in connection with the problem of rolling [3 — 6]. The main difficulty in solving such a problem is associated with the fact that the pressure cannot be sought by standard methods of inverting the Hilbert transform [7 — 9] when taking account of rheological effects. A method of reducing the viscoelastic contact problem to an elastic problem for some effective stresses and strains associated with the appropriate true values by linear differential relationships was proposed [3]. This method has been used [6] to solve problems on rolling in the presence of adhesion and sliding segments. The method of [3] does not unfortunately allow of extension to the case of several relaxation times or to the case of the presence of surface stress singularities since it results in divergent integrals in the intermediate stages of the calculations. A method for the exact solution of the viscoelastic contact problem for a bounded relaxation time spectrum by reduction of the equation for the pressure to one having the form of a Hilbert transform is proposed in [5]. However, even in this method the derivatives of improper integrals, which are divergent expressions, must be dealt with at intermediate stages of the calculations.

A simpler and more correct method of solving the sliding contact problem, which is reduced to a Riemann — Hilbert problem for a half-plane with boundary conditions of mixed type, is proposed below.

1. Model of the viscoelastic properties. Underlying the linear theory of viscoelasticity are operator equations relating the stress σ to the strain ε [10]

$$\varepsilon(t) = \mathbf{J}\sigma \equiv \int_{-\infty}^{+\infty} J(t-t')\,\sigma(t')\,dt' \tag{1.1}$$

Because of the causality principle, the Fourier transform of the kernel of the elastic aftereffect operator $J(\omega)$ (complex compliance) is analytic in the upper half-plane of the complex frequencies and has a singularity only in the lower half-plane. This property is inherent to any quantities of the type of the generalized susceptibilities [11, 12]. Assuming the singularities to be poles, we can write

$$J(\omega) = E_D^{-1}\left[1 + \sum_j \frac{a_j}{\omega_j - \omega}\right] \tag{1.2}$$

where $E_D = J^{-1}(\infty)$ is the dynamic (instantaneous) elastic modulus of the material, the ω_j determine the location of the complex compliance poles and a_j are residues at the poles. Let us note that a set of poles on the negative imaginary axis $\omega_j = 1/(i\tau_j)$ corresponds to the case of a discrete aftereffect time spectrum τ_j, and a slit along the imaginary axis to the case of a continuous spectrum. Vibrational relaxation corresponds to poles outside the imaginary axis.

It is convenient to go over to an associated reference system relative to which the stamp is fixed when finding the solution of the contact problem for a moving stamp. Under the assumption that inertial effects can be neglected, and that the pattern of the stress and strain distribution is stationary in the associated reference system, the equations for the stress and strain will have the form of the equations of classical elasticity theory with the sole difference that the integral operator of nonlocal interaction $\mathbf{J}_x$ will enter the equation in place of the reciprocal elastic modulus.

Its kernel has the Fourier transform $J_x(q) = J(-qV)$, where V is the motion velocity of the medium (we shall consider the model of a medium for which the viscoelastic properties are identical for volume and shear strains, i.e., the Poisson's ratio ν is not an operator).

If the effective stresses σ_{ij}^* are introduced according to the relationship

$$E_D^{-1}\sigma_{ij}^* = \mathbf{J}_x\sigma_{ij} \tag{1.3}$$

in place of the stresses σ_{ij}, then the equations of the customary (local) theory of elasticity will hold for the effective stresses. In application to the case under consideration of a contact problem, the fact that the effective pressure turns out to be different from zero not only within, but also outside the contact region, is a complication.

2. Solution of the contact problem. The system under consideration is shown in Fig. 1. The surface of the undeformed medium is determined by the condition $y = 0$, and the points $(\pm c, 0)$ correspond to the boundaries of the contact domain. The parameter Δ determines the position of the central axis of the cylinder with respect to the center of the contact domain. Displacements of points of the medium are given by the two dimensional vector $\mathbf{u}(x, y) = \{u_x(x, y), u_y(x, y)\}$.

In the absence of friction, finding the pressure reduces to a linear Riemann — Hilbert

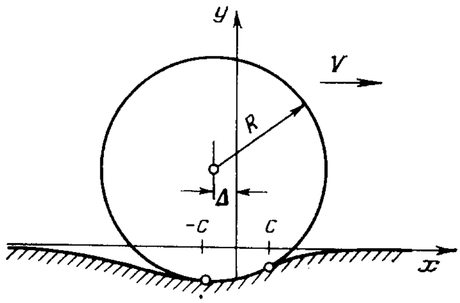

Fig. 1

conjugate problem to determine the analytic function

$$w(z)=\int\limits_{-\infty}^{+\infty}\frac{p^*(t)\,dt}{t-z}\qquad (z=x+iy)$$

in the lower half-plane of the complex variable.

For this function, the real part determined by the shape of the stamp is given in the contact domain, while the imaginary part determined by the effective pressure in this domain [9] is given outside the contact domain. According to (1.1) and (1.3), and the Fourier inversion formula (1.2), we have outside the contact domain

$$p^*(x)=\sigma_{yy}(x,0)=\begin{cases}0, & x>c\\ i\sum\limits_j\dfrac{a_j}{V}p_j\exp\left(\dfrac{i\omega_j x}{V}\right), & x<-c\end{cases}\tag{2.1}$$

$$p_j=\int\limits_{-c}^{+c}p(x)\exp\left(-\frac{i\omega_j x}{V}\right)dx=p\left(\frac{\omega_j}{V}\right)$$

where p_j is the Fourier transform of the pressure continued analytically to the point ω_j/V. The possibility of anlytic continuation results from the requirement of integrability of $p(x)$ in the contact domain.

Using standard methods of solving the conjugate problem [8, 9], and taking account of the condition $\partial u_y/\partial x=(x+\Delta)/R$ in the contact domain, we obtain for the effective pressure for $|x|<c$

$$p^*(x)=\frac{E_D}{4(1-\nu^2)R}\frac{c^2-2x(x+\Delta)}{\sqrt{c^2-x^2}}+\frac{1}{\pi\sqrt{c^2-x^2}}\times$$
$$\int\limits_{-\infty}^{-c}\frac{p^*(t)\sqrt{t^2-c^2}}{t-x}dt+\frac{A}{\pi\sqrt{c^2-x^2}}$$

From the requirement on boundedness of the strain $\partial u_y/\partial x$ at the points $x=\pm(c+0)$, the constants A and Δ can be determined, whereupon the formulas for p^* and Δ become

$$p^*(x)=\sqrt{c^2-x^2}\left[\frac{E_D}{2(1-\nu^2)R}-\frac{1}{\pi}\int\limits_{\infty}^{-c}\frac{p^*(t)}{\sqrt{t^2-c^2}\,(t-x)}dt\right]\tag{2.2}$$

$$\Delta=\frac{2(1-\nu^2)R}{\pi E_D}\int\limits_{-\infty}^{-c}\frac{p^*(x)\,dx}{\sqrt{x^2-c^2}}\tag{2.3}$$

We evaluate the Fourier transforms of the effective pressure to determine the unknown parameters $c,\ p_j$ entering into (2.2) and (2.3). Taking account of (2.1) and (2.2), an appropriate computation yields

$$p^*(q) = \int_{-\infty}^{\infty} p^*(x)\, e^{-iqx} dx = \frac{\pi E_D c}{2(1-\nu^2)R} \frac{I_1(iqc)}{iq} -$$
$$ic \sum_j \frac{\omega_j a_j p_j}{V(qV - \omega_j)} \left[I_1(iqc) K_0\left(\frac{i\omega_j c}{V}\right) + I_0(iqc) K_1\left(\frac{i\omega_j c}{V}\right)\right]$$

where $I_\nu(x), K_\nu(x)$ are Bessel functions of imaginary argument of the first and third kinds [13].

On the other hand, there follows from the definition (1.3) and the convolution theorem that

$$p^*(q) = E_D J(qV) p(q) \tag{2.4}$$

The complex compliance has zeroes at the points $\overline{\omega}_k$, where the complex modulus $E(\omega) = J^{-1}(\omega)$ has a pole. The locations of the poles of the complex modulus are determined from relaxation tests and are assumed known. It follows from (2.4) that the analytic continuation of the Fourier transform $p^*(q)$ equals zero at the points $q = \overline{\omega}_k / V$. We thereby arrive at a system of linear equations $p^*(\overline{\omega}_k/V) = 0$ to determine p_j. It is convenient to write this system as

$$\sum_j A_{kj} p_j = B_k \tag{2.5}$$
$$A_{kj} = \frac{ic}{V} \frac{\omega_j a_j}{\overline{\omega}_k - \omega_j} \left[I_1\left(\frac{i\overline{\omega}_k c}{V}\right) K_0\left(\frac{i\omega_j c}{V}\right) + I_0\left(\frac{i\overline{\omega}_k c}{V}\right) K_1\left(\frac{i\omega_j c}{V}\right)\right]$$
$$B_k = \frac{\pi E_D c}{2(1-\nu^2)R} \frac{V}{i\overline{\omega}_k} I_1\left(\frac{i\overline{\omega}_k c}{V}\right)$$

An equation determining the parameter c can be obtained from the requirement that the total pressure at the contact equal a given quantity F_y

$$F_y = \int_{-c}^{+c} p(x)\, dx = p(q)\big|_{q=0} = \left.\frac{p^*(q)}{E_D J(qV)}\right|_{q=0} = \frac{E_S}{E_D} p^*(0) \tag{2.6}$$

where $E_S = J^{-1}(0)$ is the static (creep) elastic mudulus.

The tangential component of the external force governing the resistence to slip can also be expressed in terms of $p(q)$

$$F_x = \int_{-c}^{+c} p(x) \frac{\partial u_y}{\partial x} dx = \int_{-c}^{+c} \frac{x + \Delta}{R} p(x)\, dx = \frac{1}{R}\left[i \frac{\partial p(q)}{\partial q} + \Delta \cdot p(q)\right]_{q=0}$$

By using (1.2), (2.1), (2.3), (2.4), we obtain for the friction coefficient

$$f = \frac{F_x}{F_y} = \frac{1}{R} \frac{E_S}{E_D} \left\{ \frac{1}{2} \left[\frac{4(1-\nu^2)R}{\pi E_S} - \frac{c^2}{F_y}\right] \times \right.$$
$$\left. i \sum_j \frac{a_j}{V} p_j K_0\left(\frac{i\omega_j c}{V}\right) - iV \sum_j \frac{a_j}{\omega_j^2} - \frac{c}{F_y} \sum_j \frac{a_j}{\omega_j} p_j K_1\left(\frac{i\omega_j c}{V}\right)\right\}$$

The pressure distribution function $p(x)$ can be found by means of the function

$p(q) = p^*(q) E_D^{-1} J^{-1}(qV)$ by using the inversion of the Fourier transform, however, the corresponding integral is evaluated in elementary or tabulated functions, and if necessary, can only be found numerically.

3. **Case of a standard linear body.** In the case of a standard linear, body, the rheological properties of the substance are characterized by two parameters, for which the aftereffect time τ_P and the relaxation time τ_R can be taken.

We should set

$$\omega_j = 1/(i\tau_P), \quad \overline{\omega}_k = 1/(i\tau_R)$$

in the equations written above for the passage to the standard linear body.

The other rheological parameters of the material are hence defined in terms of the characteristic times by using the relationships

$$E_D/E_S = \tau_P/\tau_R, \quad a_j = 1/(i\tau_R) - 1/(i\tau_P)$$

Equation (2.5) yields

$$p\left(\frac{1}{i\tau_P V}\right) = \frac{\pi E_D}{2(1-\nu^2)R} (\tau_P V)(\tau_R V) I_1\left(\frac{c}{\tau_R V}\right) K_1\left(\frac{c}{\tau_P V}\right) \times \left[I_1\left(\frac{c}{\tau_R V}\right) K_0\left(\frac{c}{\tau_P V}\right) + I_0\left(\frac{c}{\tau_R V}\right) K_1\left(\frac{c}{\tau_P V}\right)\right]^{-1} \quad (3.1)$$

Substituting (3.1) into (2.5), we obtain an equation to determine the size of the contact domain

$$F_y - \frac{\pi E_S c^2}{4(1-\nu^2)R} = \frac{\pi E_S c(\tau_P - \tau_R)V}{2(1-\nu^2)R} \times I_1\left(\frac{c}{\tau_R V}\right) K_1\left(\frac{c}{\tau_P V}\right)\left[I_1\left(\frac{c}{\tau_R V}\right) K_0\left(\frac{c}{\tau_P V}\right) + I_0\left(\frac{c}{\tau_R V}\right) K_1\left(\frac{c}{\tau_P V}\right)\right]^{-1} \quad (3.2)$$

which agrees with the corresponding formulas in [3, 6] to the accuracy of the notation.

4. **Conclusion.** The quantities of definite physical interest (size and location of the contact domain, friction coefficient) in the method proposed for seeking the solution of the viscoelastic contact problem of slip are determined by means of formulas with a simple algebraic structure, whose evaluation does not need numerical integration as in the method in [6]. This method can be applied to the case of vibrational relaxation, which partially takes account of the inertial properties of a viscoelastic medium, without any changes.

A specific numerical computation of the friction coefficient shows that the general nature of the dependence of the friction coefficient on the velocity with one maximum obtained in [3] is retained even for the case of the characteristic time spectra. At low velocities the resistance force is determined only by the greatest of the aftereffect times, however, near the maximum and for high velocities the terms in (1.3) corresponding to lower aftereffect times turn out to be essential. In this domain the model of a standard linear body is not a good approximation for an investigation of the velocity dependence of the friction coefficient.

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